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Propagation of a Light Ray in a Sinai Billard-Shaped Cavity: Entropic Characterization of Quasi-Regular and Chaotic Trajectories

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Abstract. Billiard dynamics has been widely studied in the literature due to its applications to model real processes. Regular behaviors (as occurs in rectangular and elliptic billiards) and disordered or chaotic behaviors (as occurs in the cases of Sinai and Bunimovich billiards, to mention just a few), have been analyzed in the literature from different approaches. In this paper we analyze the propagation of a light ray in a Sinai billiard-shaped cavity using an entropic approach that consists of calculating the image entropy in the phase space of the trajectories followed by the light ray in the billiard. The entropic difference between quasi-periodic and chaotic trajectories is studied. A one-dimensional Lorentz map is also constructed based on the distances between the initial point and the subsequent collision points at the boundaries of the billiard. The results obtained allow differentiating quasi-regular and chaotic states according to their entropic characteristics and their Lorentz maps.

Keywords: Light ray, cavity, Sinai billiard, order, chaos, entropy.

1 Introduction

Dynamics is the science that studies the variation in time of different magnitudes, i.e. their motion. There are basically three types of movements: stationary and equilibrium movements; periodic and quasi-periodic movements; and chaotic movements, the latter being those in which the prediction of the movement in a sufficiently long time is almost impossible. The word chaos and the adjective chaotic are used to describe a system that apparently has an irregular behavior and is sensitive to small changes in its initial conditions.

A clear example of this is the so-called butterfly effect, which is perhaps the most publicized analogy to imply that in chaotic dynamical systems small variations in the initial conditions can lead to unexpected results [1]. Many interesting examples of dynamical systems of problems in classical mechanics, quantum mechanics, statistics, acoustics and optics (especially those in which the interaction between particles involves elastic collisions) can be reduced to billiard systems [2].

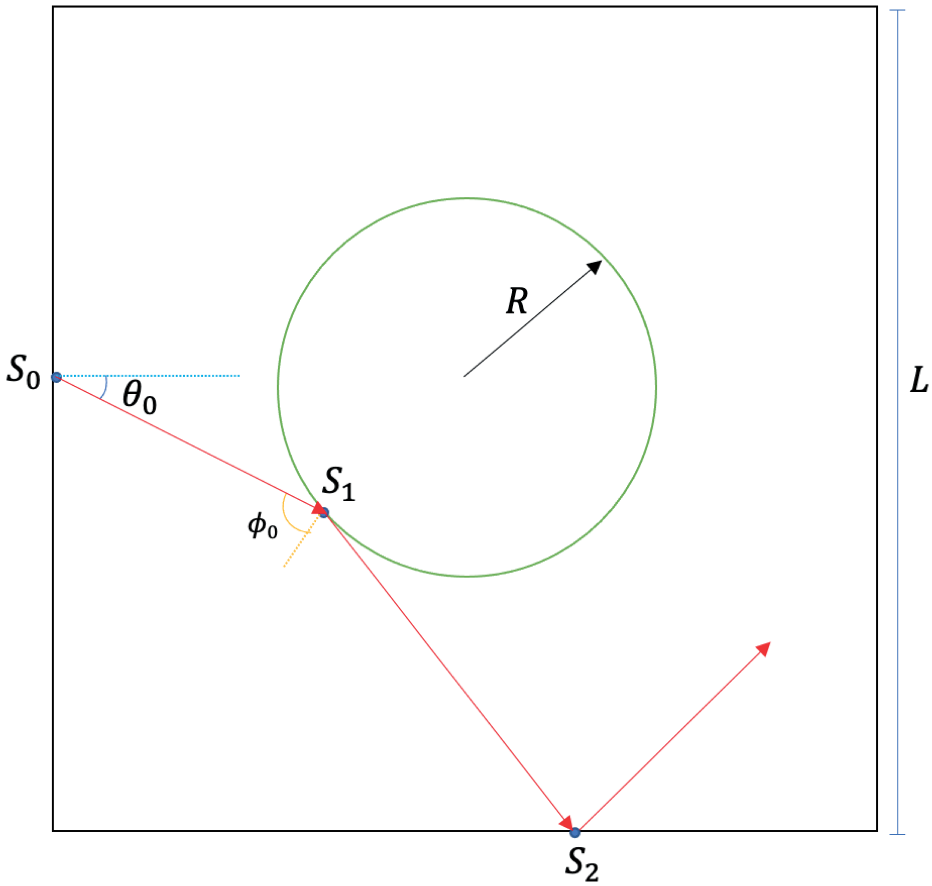

Fig. 1. The Sinai billiard consists of a square and a circular inclusion at its center. The particle moves classically experiencing only elastic collisions with the walls.

To speak of billiard systems will mean to speak of point particles moving over some region (the "billiard table") that may or may not contain smooth convex obstacles and undergoing elastic collisions against them.

2 Theory

A dynamical system representing the motion of a free particle within a bounded region of space, with elastic reflections at the boundaries, is known as a billiard [2]. The dynamical properties of billiards are determined by the shape of the boundary and can vary from totally regular to totally disordered or chaotic behaviors, similar in many respects to that of randomly evolving systems [3]. In this paper we study, in the light of classical mechanics, the Sinai billiard which is a relatively simple system presenting a strongly chaotic behavior [3].

Table 1. Table of geometries for trajectory analysis.

G_n	S_0	θ_0	$E_i(TN)$	$E_i(TC)$
G_1	$(-0, 5, 0.374)$	$\arctan(1)$	1.47511	5.17524
G_2	$(-0, 5, 0)$	$\arctan(3.5)$	4.06182	5.0533
G_3	$(-0, 5, 0)$	$\arctan(13.5)$	5.78356	5.0464
G_4	$(-0, 5, 0)$	$\arctan(32)$	5.02251	5.13676
G_5	$(-0, 5, 0)$	$\arctan(\sqrt{2}/2)$	5.03952	4.98736
G_6	$(-0, 5, 0)$	$\arctan(\sqrt{7}/2)$	2.00555	5.1196

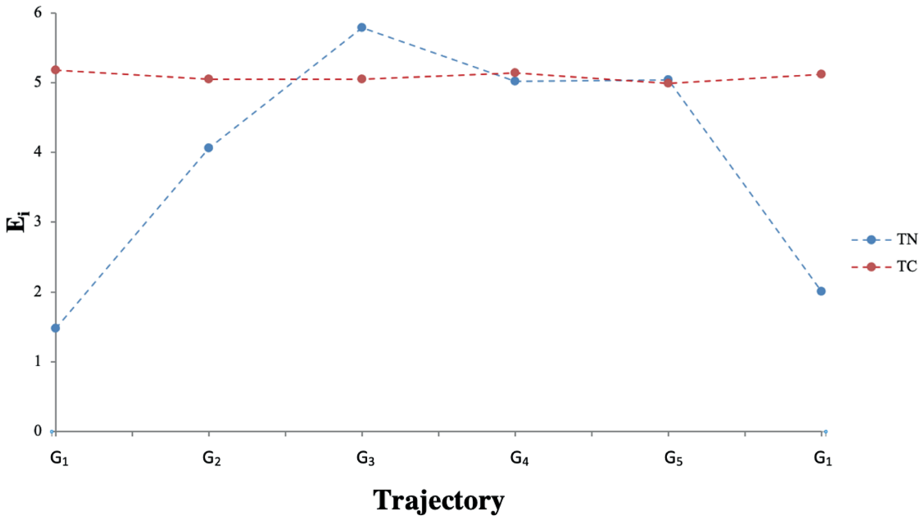

Fig. 2. Image entropy (E_i) for TN y TC.

The system consists of a particle moving in the plane inside a square (side L), at the center of which is placed a circular inclusion (radius R), as shown in Fig. 1. The particle (mass m) is assumed to obey the laws of classical mechanics and only undergoes elastic collisions against the walls of the square and the perimeter of the circular obstacle. That is, between collisions the particle moves freely following a rectilinear trajectory.

The collisions of the particle against the billiard walls are numbered consecutively by means of the index n which takes integer values, $n = 0, 1, 2, 3, \cdots$. At each collision (say, the nth) two variables are specified, namely: the position of the point where the collision occurs (variable S_n) and the angle that the direction of the movement forms with the wall immediately after the collision (angle θ_n).

We identify the position S with the parameterized distance along the perimeter measured from the lower right corner of the square. At the initial time the particle is in the state $(S_0,\, \theta_0)$, i.e., the particle undergoes a collision with the left wall at point S_0 and emerges from the collision forming an angle θ_0, as shown in Fig. 1. Knowing $(S_0,\, \theta_0)$ we want to predict the state $(S_1,\, \theta_1)$ corresponding to the next collision of the particle against a wall of the square or possibly against the circular object, whose existence depends on the values $(S_0,\, \theta_0)$ and the radius R of the disk.

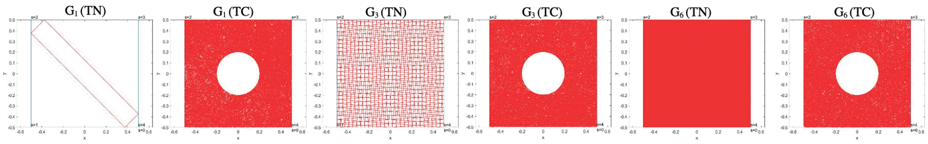

Fig. 3. Trajectories without inclusion (NT) and with inclusion (CT), for the cases G_1, G_3 and G_6 of Table 1.

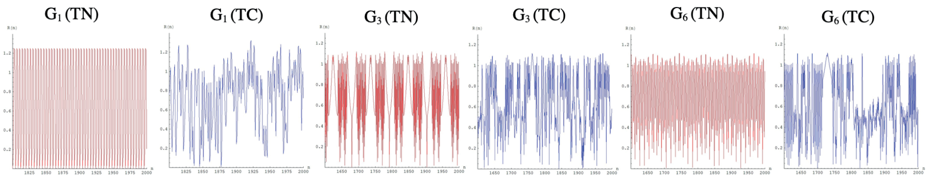

Fig. 4. Series for the trajectories without inclusion (TN) and with inclusion (TC), for the cases G_1, G_3 and G_6 of Table 1.

Once $(S_1,\ \theta_1)$ is determined, one follows with the prediction of the state $(S_2,\ \theta_2)$ and so on [4]. In the present work the point $(S_0,\ \theta_0)$ identifies the point of emission of the light ray inside the cavity. In this work, the program developed by Lansel and Porter [5] to simulate the dynamics of classical billards was used.

Given the position and direction of the previous collision, the program calculates the position and direction of the point particle after its subsequent collision with the billiard boundary. To find the location of the next collision, the program looks for an intersection between the line describing the path of the point particle and each of the collisions against the billiard walls.

Given all these intersections, the point with the minimum distance traveled is the next intersection point. To find the direction in which the point particle travels after the collision, the angle at the normal to the boundary is calculated from the derivative of the parametric equations of the billiard at the intersection point. This process leads to the mapping [4]:

$$\theta_n = 2\arctan\left(\frac{dy}{dt}\Big/\frac{dx}{dt}\right)\Bigg|_{t_n} - \theta_{n-1}, \tag{1}$$

$$\phi_n = \arctan\left(\frac{dy}{dt}\Big/\frac{dx}{dt}\right)\Bigg|_{t_n} - \theta_{n-1} + \frac{\pi}{2}, \tag{2}$$

where θ_n represents the angle with respect to the horizontal of the nth iteration, ϕ_n represents the incident angle of the nth iteration, $y(t)$ and $x(t)$ are the parametric equations of the billiard boundary, and t_n is the value of t that gives the location of the nth intersection with the boundary. The entropy is a statistical measure of randomness that can be used to characterize the texture of the input image.

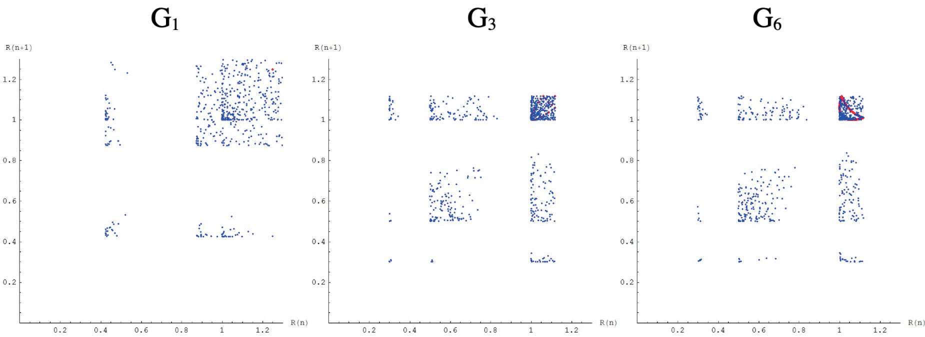

Fig. 5. Comparative plots of the Lorentz maps for G_1, G_2 and G_3. The red and blue points correspond to the maps for the cases TN and TC, respectively.

We use the Mathematica program to calculate the image entropy E_i, defined by:

$$E_i = -\sum_{k=1}^{N} p_k \log_2 (p_k) \,, \tag{3}$$

where N is the total number of pixels of the image and p_k is the probability value associated with the grayscale level of the kth pixel. The distances between the emission point S_0 and the collision points S_n $(n = 1, 2, 3, ...)$ are the parameters that define a series as a function of the collision number as follows:

$$R_n = \overline{S_0\, S_n}. \tag{4}$$

Lorentz maps are constructed for the R_n series of Eq. (3).

3 Results

Table 1 shows the analyzed trajectories G_n, where 2000 iterations were taken in each case. The characteristics of the light rays, initial position S_0 and initial angle θ_0, as well as the image entropies E_i, both for the trajectories with absence (TN) and presence (TC) of circular inclusion, are shown in the table. Figure 2 shows the entropic behavior. In Fig. 3 the TN and TC trajectories for a circular inclusion of $R = 0.2$ are shown, for some of the cases in Table 1 (G_1, G_3 and G_6). Figure 4 presents the series associated with the distances between the light ray emission point and the collision points as a function of the collision number. Figure 5 shows the Lorentz maps for the TN and TC trajectories, for some of the cases presented in Table 1.

4 Conclusions

The dynamics of the propagation of a light ray in a Sinai billiard-shaped cavity for the TN and TC trajectories was studied. The calculation of the image entropies E_i and the construction of Lorentz maps allowed us to identify characteristics of the dynamics for the TN and TC trajectories. In particular, the Lorentz maps graphically showed regions where the quasi-regular and chaotic nature of the TN and TC trajectories can be easily recognized. The results of this work can be applied in the design and construction of two-dimensional light traps.

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